Linear Algebra I

08/11/2024, Friday, 15:00 – 17:00

You are **NOT** allowed to use any type of calculators.

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1 Subspaces of $\mathbb{R}^n$ — $5 + 10 + 10 = 25$ pts

(a) Let $\lambda \in \mathbb{C}$ and $A \in \mathbb{R}^{n \times n}$. Show that the set $\{x \in \mathbb{R}^n \mid Ax = \lambda x\}$ is a subspace.

(b) Let X and Y be two subspaces of $\mathbb{R}^n$.

  (i) Suppose that $X \not\subseteq Y$. Show that if $X \cup Y$ is a subspace, then $Y \subseteq X$.

  (ii) Show that $X \cup Y$ is a subspace if and only if $X \subseteq Y$ or $Y \subseteq X$.

2 Diagonalization — $4 + 4 + 5 + 6 + 6 = 25$ pts

Consider the matrix

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

(a) Is M diagonalizable? Justify your answer.

(b) Is M unitarily diagonalizable? Justify your answer.

(c) Find the eigenvalues of M.

(d) Find a diagonalizer for M.

(e) Solve the difference equation

$$\boldsymbol{x}_{k+1} = M\boldsymbol{x}_k \quad \text{where} \quad \boldsymbol{x}_0 = \begin{bmatrix} 6 \\ 12 \\ 12 \end{bmatrix}.$$

3 Eigenvalues/eigenvectors $5 + 5 + 5 + 5 = 20$ pts

Suppose that $N \in \mathbb{F}^{n\times n}$ is a *nilpotent* matrix, that is $N^k = 0$ for some $k \geqslant 0$. Let a be a scalar.

(a) Find the eigenvalues of N.

(b) Show that $aI_n + N$ is nonsingular if and only if $a \neq 0$.

(c) Find the characteristic polynomial of $aI_n + N$.

(d) Find the determinant and trace of $aI_n + N$.

4 Gram-Schmidt process $8 + 2 + 10 = 20$ pts

Let $n \geqslant 3$ and $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$ be vectors in $\mathbb{F}^n$ satisfying

$$\begin{gathered}\|\boldsymbol{x}_1\| = 1\\ \|\boldsymbol{x}_2\| = \|\boldsymbol{x}_3\| = 2\\ \boldsymbol{x}_1^*\boldsymbol{x}_2 = \boldsymbol{x}_2^*\boldsymbol{x}_3 = \boldsymbol{x}_3^*\boldsymbol{x}_1 = 1.\end{gathered}$$

(a) Show that $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$ are linearly independent.

(b) Is the set $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$ orthonormal?

(c) By using the Gram-Schmidt process, find an orthonormal set $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3\}$ such that

$$\operatorname{span}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) = \operatorname{span}(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3).$$

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10 pts free